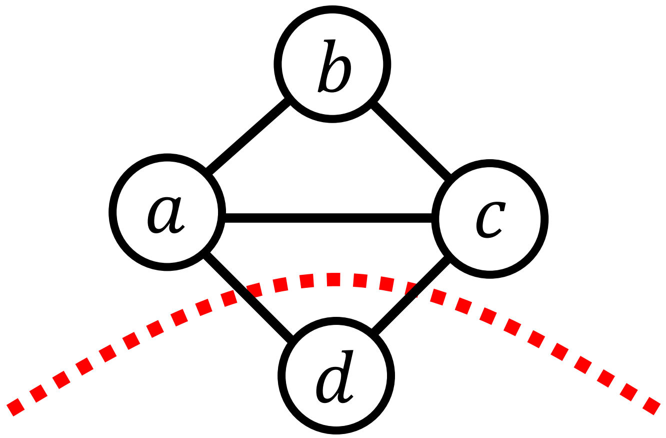
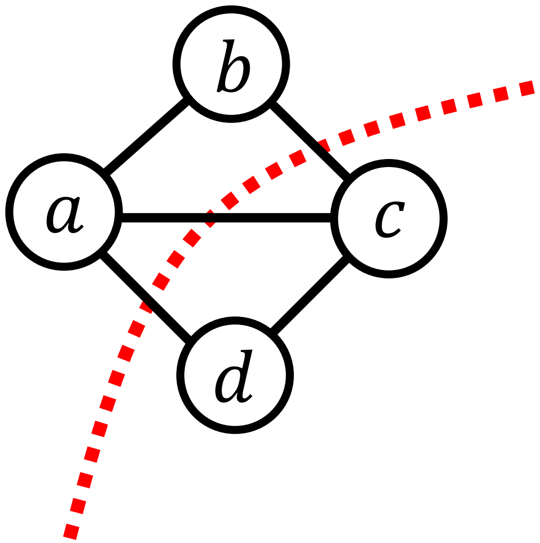
# Workshop #2
## CSCI 432

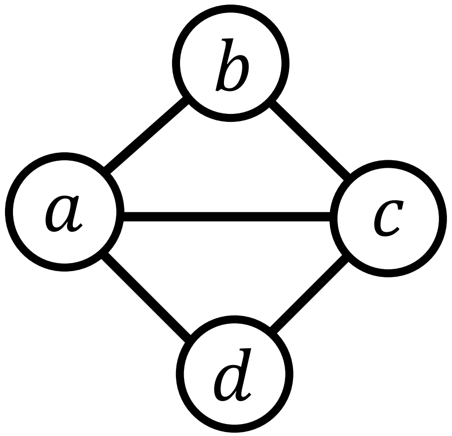
# Global Minimum Cut

<u>Global Cut:</u> Given an undirected graph $G = (V, E)$, a *cut* is a partition of vertices into two disjoint subsets. The *size* of the cut is the number edges that cross the cut (i.e., the number of edges with an endpoint in each set).



# Contraction Algorithm



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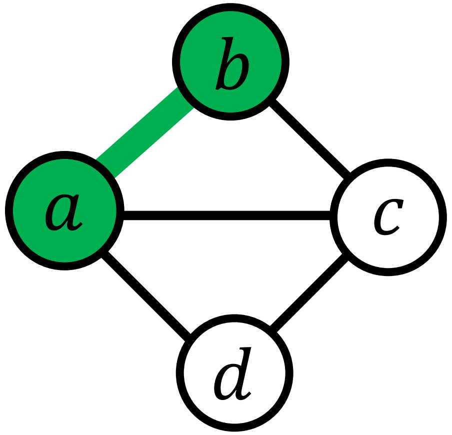


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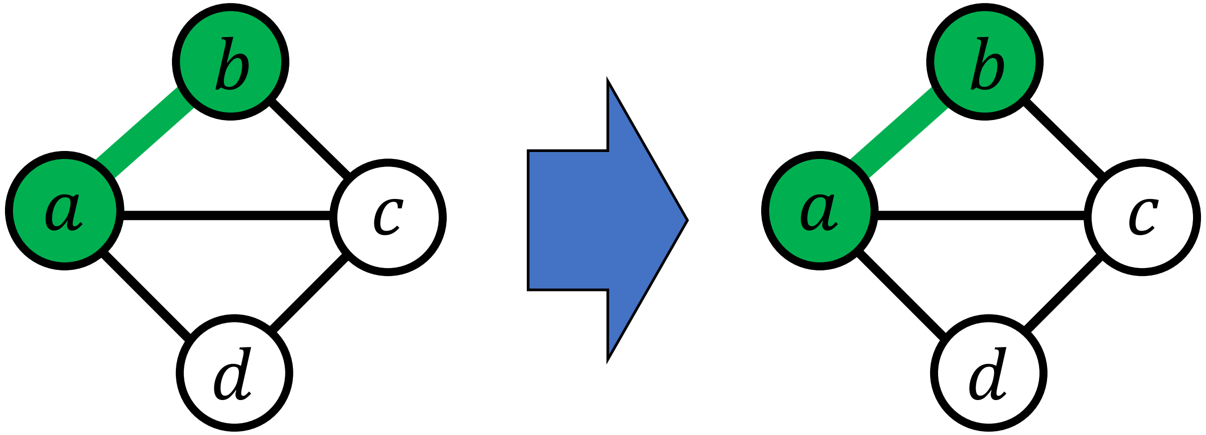


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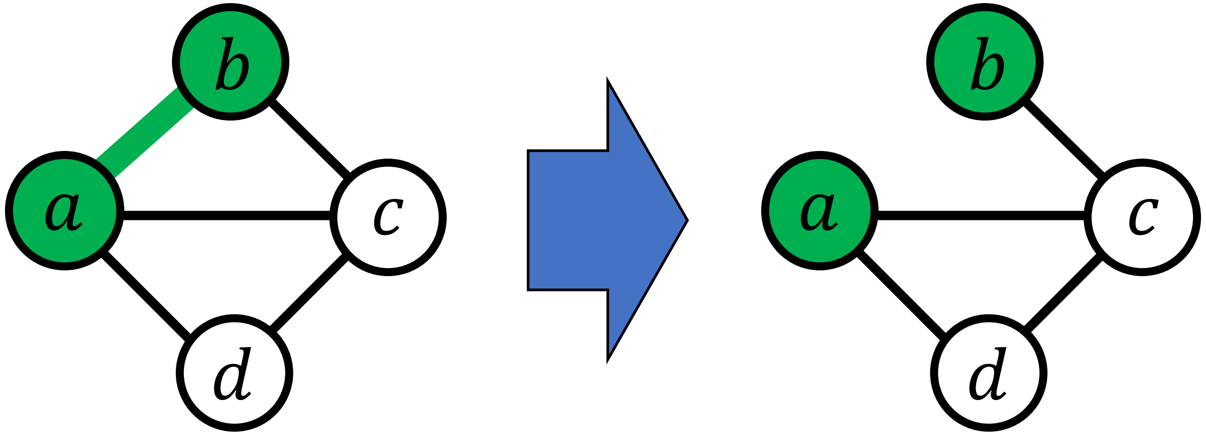


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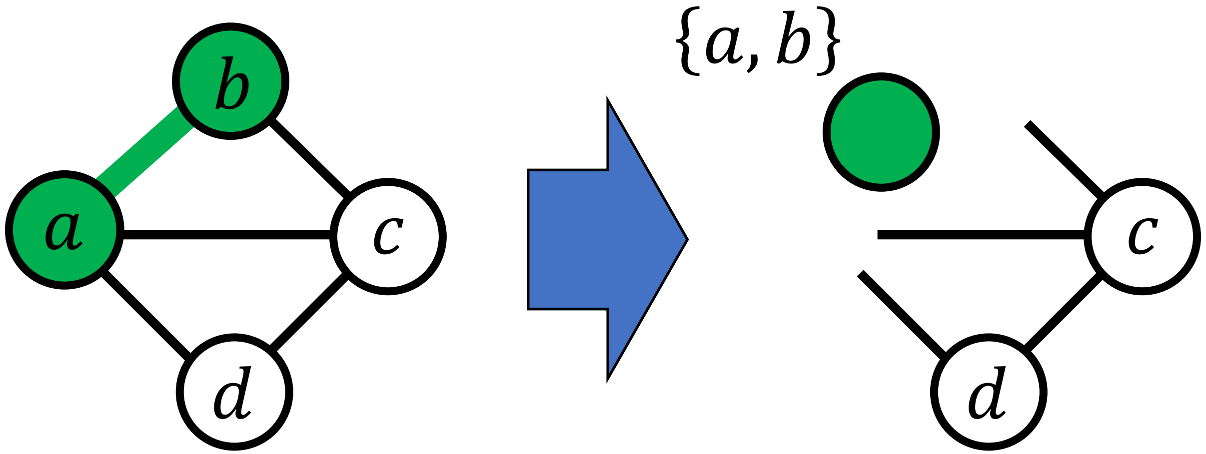


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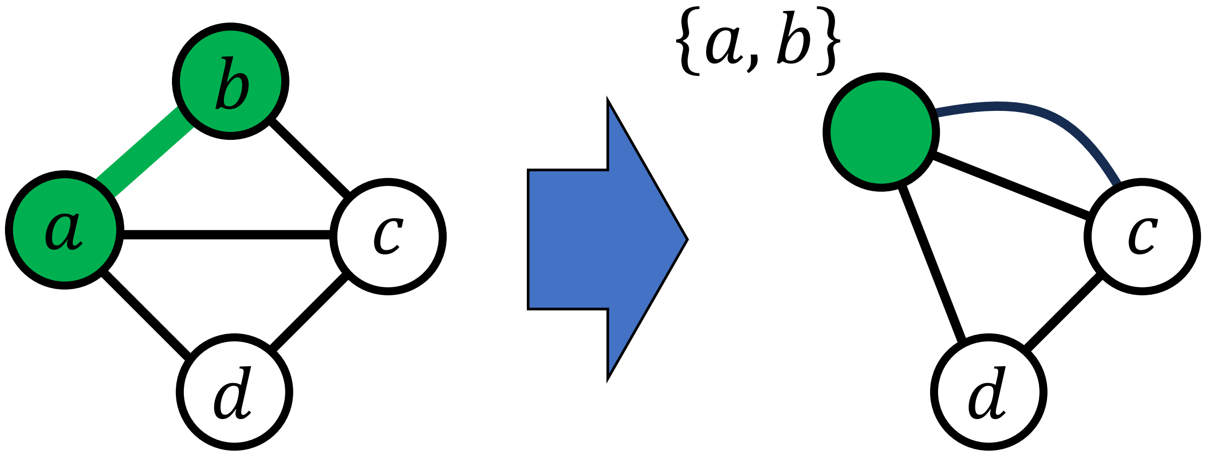


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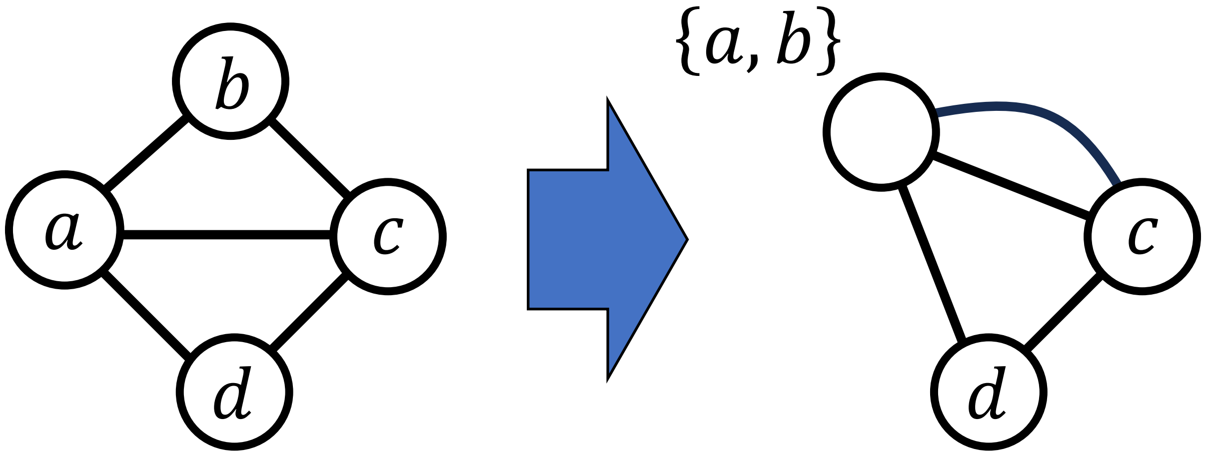


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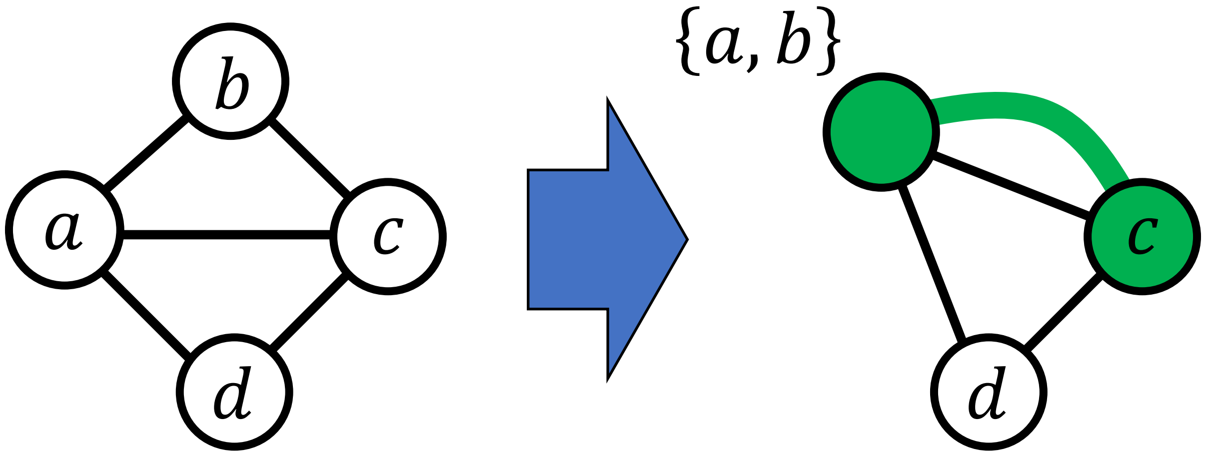


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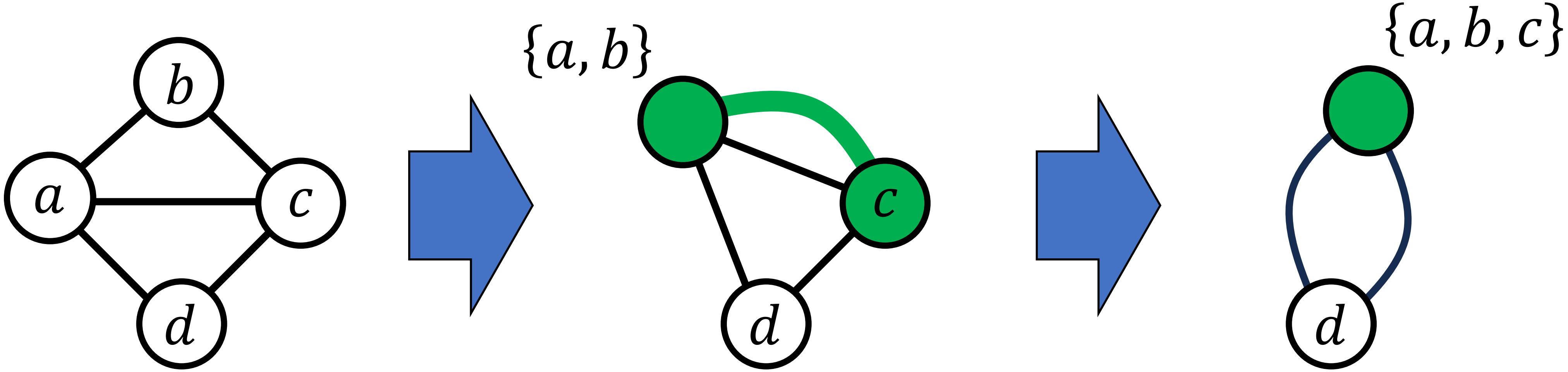


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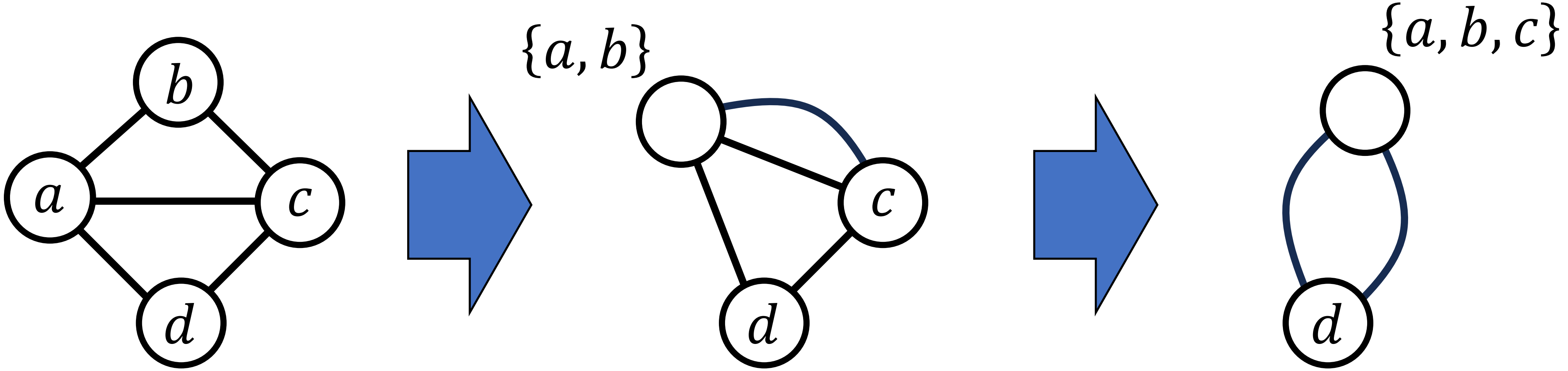


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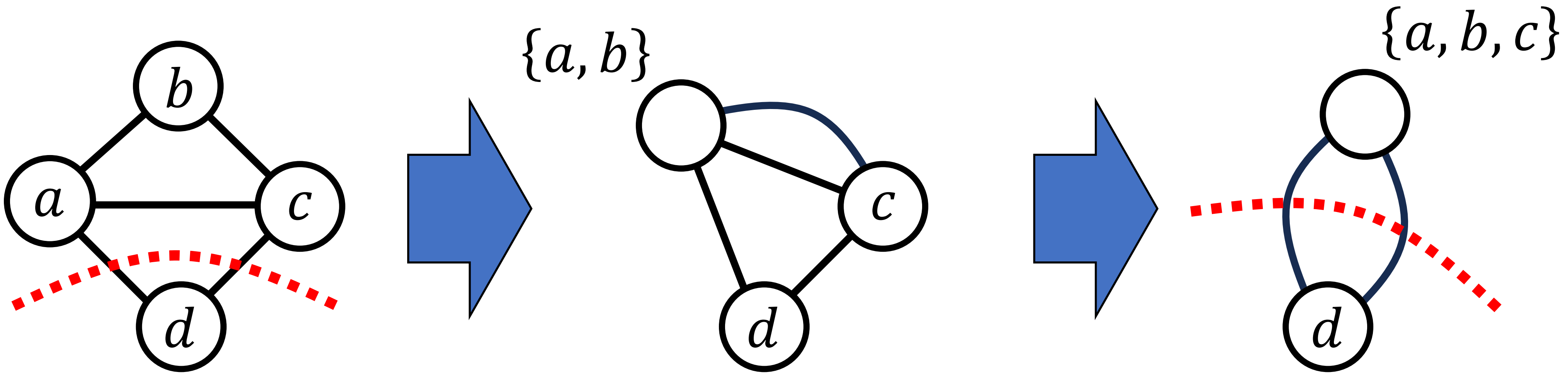


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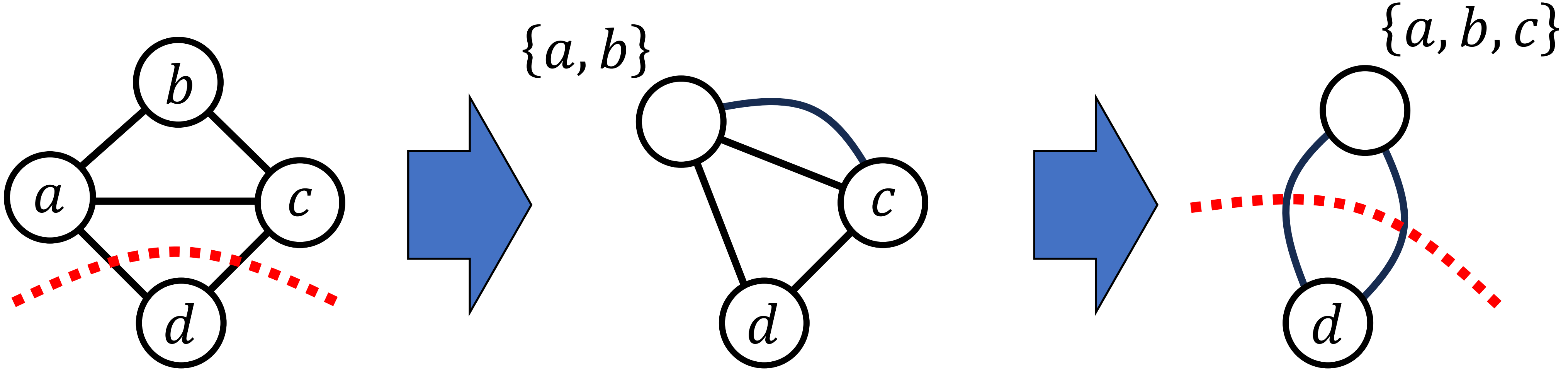
**Find a series of contractions that do not yield an optimal solution:**

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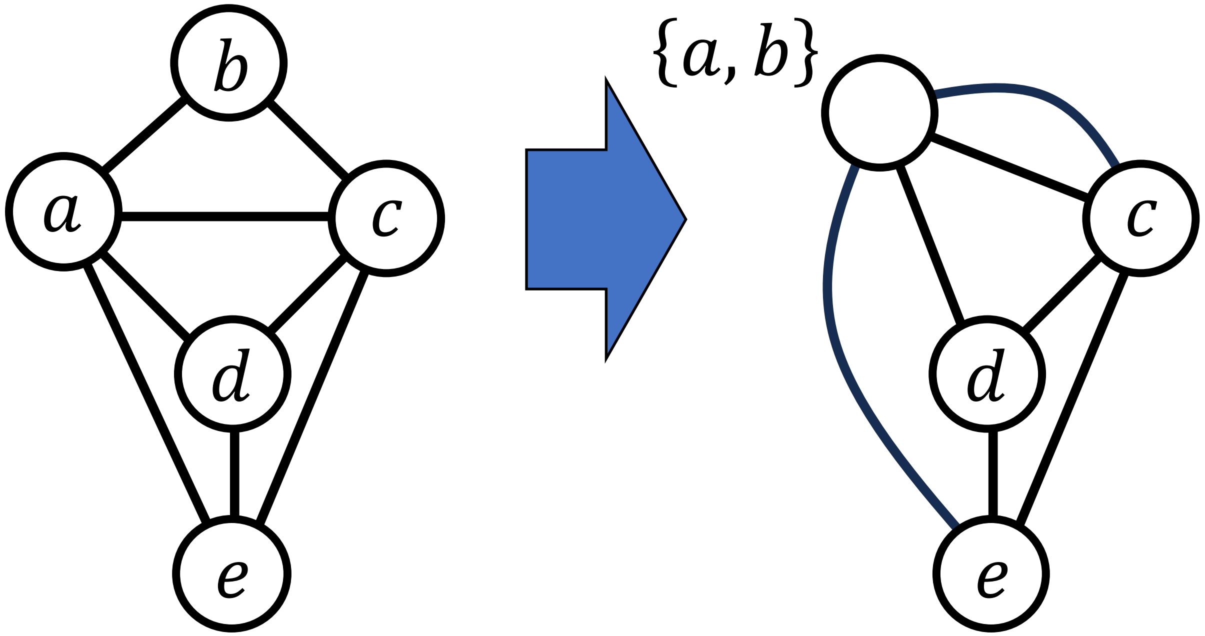

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# Probability of Succeeding

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# Probability of Succeeding

Suppose $(A, B)$ is a minimum cut of size $k$. Let $F$ be the set of edges between $A$ and $B$.

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$$\Pr[\varepsilon_1 \cap \varepsilon_2 \cap \cdots \cap \varepsilon_{n-2}] = ??$$

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$$\geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right) \cdots \left(1 - \frac{2}{3}\right)$$

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$$= \left(\frac{\cancel{n-2}}{n}\right)\left(\frac{\cancel{n-3}}{n-1}\right)\left(\frac{\cancel{n-4}}{\cancel{n-2}}\right)\left(\frac{\cancel{n-5}}{\cancel{n-3}}\right)\left(\frac{\cancel{n-6}}{\cancel{n-4}}\right) \ldots \left(\frac{2}{\cancel{4}}\right)\left(\frac{1}{\cancel{3}}\right)$$

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$$\begin{aligned}
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&\geq \left(1 - \frac{2}{n}\right)\left(1 - \frac{2}{n-1}\right) \cdots \left(1 - \frac{2}{3}\right) \\
&= \left(\frac{n-2}{n}\right)\left(\frac{n-3}{n-1}\right)\left(\frac{n-4}{n-2}\right)\left(\frac{n-5}{n-3}\right)\left(\frac{n-6}{n-4}\right) \ldots \left(\frac{2}{4}\right)\left(\frac{1}{3}\right) \\
&= \frac{2}{n(n-1)}
\end{aligned}$$

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| $n$ | Pr |
|---|---|
| 10 | 0.022 |
| 25 | 0.0033 |
| 50 | 0.00082 |

# Probability of Succeeding

Suppose $(A, B$ … dges

between $A$ ar …


Let $\varepsilon_j =$ Event …

$\Pr[\varepsilon_1 \cap \varepsilon_2 \cap \cdots$ … $\cap \cdots \cap \varepsilon_{n-3}]$

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$$\geq \left(1-\frac{2}{n}\right)\left(1-\frac{2}{n-1}\right)\cdots\left(1-\frac{2}{3}\right)$$

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